MODELS OF NARROW-BAND HARMONIC PROCESSES

In this paper simple models of harmonic processes applied in the signal theory have been analyzed. Formula (8), which describes the amplitude density function of the narrow-band uniform process being a strictly stationary process, was introduced. Furthermore, the form of the density function of the sum of the harmonic process and the uniformly distributed random variable was derived (equation (16)).

KEYWORDS: harmonic normal process, harmonic uniform process

1. INTRODUCTION

In the wake of the development of mathematical methods describing more and more complex problems involved in technical sciences, stochastic processes are playing more and more important role in problem solving within the range of engineering practice.

Paper [3] provides numerous examples of applying stochastic dynamics of technical units:
- stochastic vibrations of road vehicles excited by random road surfaces;
- stochastic vibrations of elastic boards excited by turbulent input;
- response of a construction to seismic input;
- response of craft and offshore platforms to sea wave input;
- stochastic stability of construction.

Furthermore, stochastic processes have found application in control system analysis in the presence of random noise, aerodynamics, population dynamics, pollutant transport in groundwater.

Stochastic processes have found multiple applications in telecommunications [3, 4]. It results from a vast array of signals involved in communications issues. This, in turn, renders deterministic signal models useless and necessitates the use of stochastic signal models for description.

In this paper, the simplest possible model applied in the signal theory was considered, namely, harmonic normal process with random phase. The process was subsequently modified replacing the normal distribution with the uniform distribution. For this process an analytic form of amplitude density function was

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derived (equation (8)). Furthermore, the sum of the stochastic process and
the uniformly distributed noise was analyzed obtaining the analytic form
of the amplitude density function (equation (16)).

2. NARROW–BAND HARMONIC PROCESS

2.1. Narrow-band normal process

The following process is considered [2, p.40]

\[ x(t) = a \cdot \sin(\omega_0 t + \varphi) + \alpha \cdot \sin(\omega_0 t) + \beta \cdot \cos(\omega_0 t) \] (1)

It is assumed that \( \alpha \) and \( \beta \) are random variables and follow the normal
distribution \( N(0, \sigma) \).

From equation (1), results the following:

\[ \alpha = a \cos \varphi \] (2)
\[ \beta = a \sin \varphi \]

On the basis of equation (2) it can be observed that random variables \( a \) and \( \varphi \)
fulfil dependencies:

\[ a = \sqrt{\alpha^2 + \beta^2} \] (3)
\[ \varphi = \arctg \frac{\beta}{\alpha} \]

It can be shown, e.g. in [4, p.66], that random variable \( \varphi \) is uniformly
distributed:

\[ f(\varphi) = \frac{1}{2\pi} \text{ dla } |\varphi| < \pi \] (4)

whereas variable \( a \) is described by the Rayleigh distribution

\[ f(a) = \frac{a}{\sigma^2} \exp \left( -\frac{a^2}{2\sigma^2} \right) \] (5)

Figure 1 shows theoretical dependencies \( f(a) \) (equation (5)) and \( f(\varphi) \) (equation
(4)) marked with a solid line. The dotted line (\( f(a \) and \( f(\varphi \) ) shows the results of
computer simulations using a random number generator following the normal
distribution for \( N = 100000 \) repetitions.

The harmonic normal process can be treated as a border of a narrow-band
filtration of narrow-band noise when the bandwidth approaches zero. It can be
the result of radio transmission of harmonic signal dispersed by several object
reflections as well. [4, p.67].
2.2. Uniformly distributed narrow-band process

Again the process that satisfies (1) is considered, assuming that \( \alpha \) and \( \beta \) are uniformly distributed random variables \( \alpha \in (-B, B) \), \( \beta \in (-B, B) \). In order to determine the density function of variable \( a \), one has to determine the density function of variable \( y = g(\alpha) \) where \( g(\alpha) = \alpha^2 \) using formula [1, p.140]

\[
f_y(y) = \sum_{i=1}^{n} \frac{f_\alpha(\alpha_i)}{|g'(\alpha_i)|}
\]  

(6)

where \( f_\alpha(\alpha) \) the density function of random variable \( \alpha \); \( g'(\alpha) = 2\alpha \); \( \alpha_i \) – roots of equation \( y = g(\alpha) \), i.e. \( \alpha_1 = \sqrt{y} ; \alpha_2 = -\sqrt{y} \).

Similarly one determines the density function of variable \( z = g(\beta) \) where \( g(\beta) = \beta^2 \).

Subsequently, the density function of variable \( w = y + z \) is calculated based on dependency [1, p.205]

\[
f_w(w) = \int_{-\infty}^{\infty} f_y(w-z)f_z(z)dz = \int f_y(y)f_z(w-y)dy
\]  

(7)

Finally, applying (6), the density function of variable \( a = \sqrt{w} \) is determined obtaining:

\[
f(a) = \begin{cases} 
2\pi a & \text{dla } a < B \\
4a \left(\arctg \frac{B}{\sqrt{a^2-B^2}} - \arctg \frac{\sqrt{a^2-B^2}}{B}\right) & \text{dla } a > B
\end{cases}
\]  

(8)
Figure 2 (chart on the left) shows theoretical dependency $f_{\alpha\alpha}$ (equation (8)) marked with a solid line and the results of computer simulations $f_{\alpha\alpha}$ are marked with a dotted line. The chart on the right shows the empirical density function $f_{\psi\psi}$ of random variable $\psi$.

![Chart showing theoretical and empirical density functions](chart.png)

While some similarity between the density function $f_{\alpha\alpha}$ in Figure 1 and 2 can be observed, the density functions of random variable $\psi$ vary substantially. In the case of normally distributed narrow-band process, variable $\psi$ follows uniform distribution. As shown in Figure 2, for a uniformly distributed narrow-band process, the maximum value of density function $f_{\psi\psi}$ occurs when $\psi = \pm \frac{\pi}{4}$ and $\psi = \pm \frac{3\pi}{4}$ and minimum occurs when $\psi = \pm \frac{\pi}{2}$. The value of parameter $B$ determining the uniform function does not influence the form of the density functions presented in Figure 2.

2.3. Laplace distributed narrow-band process

Similarly as in subsection 1.1, the process that satisfies (1) is considered, assuming that $\alpha$ and $\beta$ are Laplace distributed random variables:

$$f(\alpha) = \frac{\lambda}{2} \exp(-\lambda |\alpha|) \quad f(\beta) = \frac{\lambda}{2} \exp(-\lambda |\beta|)$$

(9)
By means of computer simulations and application of equation (3), the values of random variables $a$ and $\psi$ were determined. Figure 3 shows the results of a numerical experiment for parameter $\lambda = 0.4$.

Fig. 3. Laplace distributed narrow-band process (for parameter $\lambda = 0.4$). Chart on the left-hand side presents density function $f_a$ of random variable $a$. Chart on the right-hand side presents empirical density function of random variable $\psi$.

The value of parameter $\lambda$ (equation (9)) does not substantially influence the density function $f_a$ presented in Figure 4. However it substantially influences the form of density function $f_\psi$. Namely, in points $\psi = \pm \frac{\pi}{2}$ occur: function minimum for parameter $\lambda = 0.4$ function maximum for parameter $\lambda = 1$ and $\lambda = 8$.

Fig. 4. Laplace distributed narrow-band process. Density function $f_\psi$ (left-hand side) corresponds to parameter value $\lambda = 1$. Right-hand side shows the density function of random variable $\psi$ for parameter value $\lambda = 8$.
3. HARMONIC SIGNAL DISTURBED BY UNIFORMLY DISTRIBUTED NOISE

The harmonic process with random phase is given by:

\[ x(t) = A \cdot \sin(\omega_0 t + \psi) \]  

where \( \psi \) is a uniformly distributed random variable described by formula (4).

The process was disturbed by a random variable following the uniform distribution \( z \in (-B, B) \)

\[ f_z(z) = \frac{1}{2B} \quad \text{for } |z| < B \quad \text{and} \quad f_z(z) = 0 \quad \text{for } |z| > B \]  

One can focus on the probability distribution of random variable \( y = x + z \)

The harmonic process with random phase is an example of a strictly stationary ergodic process, the density function of which is given by [4, p.63]

\[ f_x(x) = \frac{1}{\pi \sqrt{A^2 - x^2}} \quad \text{for } |x| < A \quad \text{and} \quad f_x(x) = 0 \quad \text{for } |x| > A \]  

From equations (7) and (11), one can obtain:

\[ f_y(y) = \frac{1}{2B} [F_x(y + B) - F_x(y - B)] \]  

where:

\[ F_x(x) = \int_{-A}^{x} f_x(t) dt = \frac{1}{2} + \frac{1}{\pi} \arcsin \frac{x}{A} \]  

From equations (14) and (15) one can obtain the following form of the density function of variable \( y \):

\[ f(y) = \begin{cases} G(y) & \text{for } B < 2A \\ g(y) & \text{for } B > 2A \end{cases} \cdot P(y) \]  

where:

\[ P(y) = \begin{cases} 0 & \text{for } |y| > A + B \\ 1 & \text{for } |y| < A + B \end{cases} \]

\[ g(y) = \begin{cases} \frac{1}{2B} & \text{for } y > A - B \text{ and } y < B - A \\ G_3(y) & \text{for } y > A - B \text{ and } y > B - A \end{cases} \]
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\[
G(y) = \begin{cases} 
G_1(y) & \text{for } y < B - A \\
G_2(y) & \text{for } y > B - A \text{ and } y < A - B \\
G_3(y) & \text{for } y > B - A \text{ and } y > A - B 
\end{cases}
\]

\[
G_1(y) = \frac{1}{2B} \left( \frac{1}{\pi} \arcsin \frac{y + B}{A} + \frac{1}{2} \right)
\]

\[
G_2(y) = \frac{1}{2B} \left( \frac{1}{\pi} \arcsin \frac{y - B}{A} \right)
\]

\[
G_3(y) = \frac{1}{2B} \left( \frac{1}{\pi} \arcsin \frac{y}{A} - \frac{1}{\pi} \arcsin \frac{y - B}{A} \right)
\]

Figure 5 shows theoretical dependency $f_t$ (equation (16)) marked with a solid line, and the result of density function $f_c$ computer simulations of variable $y$ ($B = 1$) marked with a dotted line. The chart on the left-hand side was made for amplitude $A = 0.05$, and the right-hand side for $A = 0.5$.

Fig. 5. Result of the disturbance of a harmonic process with a random phase by uniformly distributed noise. Solid line presents theoretical dependency $f_t$ (equation (16)) and dotted line presents the results of density function computer simulations $f_c$ of variable $y$ ($B = 1$). The chart on the left-hand side was made for amplitude $A = 0.05$, and the right-hand side for $A = 0.5$.

Theoretical curve $f_{t1}$ bears a close resemblance to the density function of the uniform distribution. It stems from the fact that the random variable interval $<-1,1>$ was substantially larger than the harmonic process interval $<-0.05,0.05>$. In the case of curve $f_{t2}$, made for $A = 0.5$ and $B = 1$, in the central part one can notice the trace of the density function of the uniform distribution.

The theoretical curve $f_{t3}$ represents the case when the random variable interval and the harmonic process interval are identical and equal $<-1,1>$. The curve $f_{t4}$ coincides almost ideally with the density function curve of the
harmonic process described by equation (13). It stems from the fact that amplitude $A = 20$ is many times larger than parameter $B = 1$.

![Graph](image)

**Fig. 6.** Result of the disturbance of a harmonic process with a random phase by uniformly distributed noise ($B = 1$). Labeling is the same as in Figure 5. Curves $f_3$ and $f_{i3}$ correspond to amplitude $A = 1$, and curves $f_4$ and $f_{i4}$ were obtained for amplitude $A = 20$.

**CONCLUSIONS**

In section 1 a couple of models of narrow-band harmonic process were considered. In subsection 1.2 formula (8) was derived, which describes the density function of amplitude of uniformly distributed narrow-band process. On comparison of Figures 1 and 2, one can notice certain similarity between the density function of amplitude of uniformly distributed narrow-band process and the density function of amplitude of normal process described by the Rayleigh distribution (equation (5)). However, phases of both processes differ fundamentally. For the normal process, the random variable follows uniform distribution.

In the case of the uniformly distributed narrow-band process, for the density function $f_\psi$ the maximum occurs for $\psi = \pm \frac{\pi}{4}$ and $\psi = \pm \frac{3\pi}{4}$ and the minimum for $\psi = \pm \frac{\pi}{2}$. Similarly to the normally distributed process, the uniformly distributed process is strictly stationary – the formulas describing amplitude density functions (8) and (13) do not contain a time parameter $t$ variable.

In section 2 the case was considered when the harmonic signal with a random phase was disturbed by a uniformly distributed noise (equation (16)).
REFERENCES


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